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Modulability and duality of certain cones in pluripotential theory

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ABSTRACT

Let $p > 0$, and let $\mathcal{E}_p$ denote the cone of negative plurisubharmonic functions with finite pluricomplex p-energy. We prove that the vector space $\delta\mathcal{E}_p = \mathcal{E}_p - \mathcal{E}_p$, with the vector ordering induced by the cone $\mathcal{E}_p$ is σ-Dedekind complete, and equipped with a suitable quasi-norm it is a non-separable quasi-Banach space with a decomposition property with control of the quasi-norm. Furthermore, we explicitly characterize its topological dual. The cone $\mathcal{E}_p$ in the quasi-normed space $\delta\mathcal{E}_p$ is closed, generating, and has empty interior.

1. Introduction

The theory of ordered vector spaces is a fundamental tool in functional analysis with a wide variety of applications in for example engineering and economics (see e.g. [8,19,23,31]). In this article we shall make use of the theory of ordered vector spaces and apply it to pluripotential theory. We now continue with a brief discussion about the setting, and we refer the reader to Sections 2 and 3 for more detailed background and definitions. Let A be an open set in $\mathbb{C}^n$. An upper semicontinuous function $u : A \to \mathbb{R} \cup \{-\infty\}$ is called *plurisubharmonic* if the Laplacian of u, in the sense of distributions, is non-negative along each complex line that intersects A, in other word, u is plurisubharmonic if it is subharmonic on each complex line that intersects A. The function that is identically $-\infty$ is not plurisubharmonic by fiat. The family of plurisubharmonic functions defined A is denoted by $\mathcal{PSH}(A)$. Throughout this article we always assume that a plurisubharmonic function is defined on a so-called hyperconvex domain $\Omega \subset \mathbb{C}^n$. This assumption is made to ensure a satisfying amount of plurisubharmonic functions with certain properties. The most important objects in this article are the cones $\mathcal{E}_p \subset \mathcal{PSH}(\Omega)$. Let $p > 0$, then $\mathcal{E}_p$ is defined to be the family of all negative plurisubharmonic functions with well-defined, and finite, pluricomplex p-energy. These cones were introduced and studied in [12] (see also [3,15,21,33]). It is not only within pluripotential theory these cones have been proven useful, but also as a tool in dynamical systems and algebraic geometry (see e.g. [2,18]).

In this article, we want at first to consider, $\mathcal{E}_p$ in a vector space. The natural candidate is $\mathcal{E}_p - \mathcal{E}_p$. To simplify the notations we therefore set $\delta\mathcal{E}_p = \mathcal{E}_p - \mathcal{E}_p$. This construction yields that $\mathcal{E}_p$ is generating in $\delta\mathcal{E}_p$. By some straightforward calculations it is noted that $\delta\mathcal{E}_p$ is a vector space under pointwise addition and usual scalar multiplication. One way of dealing with the situation $-\infty - (-\infty)$ in this vector space is to implement the convention $-\infty - (-\infty) = -\infty$. Furthermore, if we equip $\delta\mathcal{E}_p$ with the vector ordering induced by the cone $\mathcal{E}_p$, then it is a σ-Dedekind complete ordered vector space

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(Theorem 3.2). A simple example shows that in $\delta\mathcal{E}_p$, the vector ordering induced by the cone $\mathcal{E}_p$ does not coincide with the classical pointwise ordering (Example 3.1).

Let $p > 0$. Then for $u \in \delta\mathcal{E}_p$ we define:

$$\|u\|_p = \inf_{\substack{u_1 - u_2 = u \\ u_1, u_2 \in \mathcal{E}_p}} \left(\int_\Omega \left(-(u_1+u_2)\right)^p \left(dd^c(u_1+u_2)\right)^n \right)^{\frac{1}{n+p}}, \tag{1.1}$$

where $(dd^c\cdot)^n$ is the complex Monge–Ampère operator. If $p = 0$, then we shall use (1.1) with the convention that $(-(u_1+u_2))^p = 1$. Our aim in Section 4 is to prove that $(\delta\mathcal{E}_p, \|\cdot\|_p)$ is a quasi-Banach space, and for $p = 1$ a Banach space (Theorem 4.7). As a direct consequence we get that $\mathcal{E}_p$ is closed in $(\delta\mathcal{E}_p, \|\cdot\|_p)$ (Corollary 4.8), and has empty interior (Theorem 4.9).

The aim of Section 5 is to prove the following theorem.

Theorem 5.4. *For $p > 0$ we have that*

(1) *the cone $\mathcal{E}_p$ is a normal in $(\delta\mathcal{E}_p, \succcurlyeq, \|\cdot\|_p)$, and*
(2) *the dual cone $\mathcal{E}_p'$ is generating in $(\delta\mathcal{E}_p, \succcurlyeq, \|\cdot\|_p)'$, i.e. $(\delta\mathcal{E}_p)' = \mathcal{E}_p' - \mathcal{E}_p'$.*
(3) *Furthermore, for $p \geqslant 1$ the dual space $(\delta\mathcal{E}_p, \succcurlyeq, \|\cdot\|_p)'$ is equal to the closure of $\delta\mathcal{M}_p$ in $\sigma((\delta\mathcal{E}_p)', \delta\mathcal{E}_p)$, where $\sigma((\delta\mathcal{E}_p)', \delta\mathcal{E}_p)$ is usual weak*-topology of $(\delta\mathcal{E}_p)'$, and*

$$\mathcal{M}_p = \left\{\mu\colon\ \mu \text{ is a positive Radon measure such that } \left(dd^c u\right)^n = \mu \text{ for some } u \in \mathcal{E}_p\right\}.$$

Let $\mathcal{K}$ be a generating cone in a vector space X, i.e. every $u \in X$ can be written as $u = u_1 - u_2$, $u_1, u_2 \in \mathcal{K}$. This decomposition is not unique. It is a classical problem going back to Bonsall [9], Grosberg and Krein [20], and Pierce [34], to find suitable decompositions. We prove that there exists a decomposition of each element in $\delta\mathcal{E}_p$ with control of the quasi-norm (Theorem 6.2). As an application of this decomposition theorem we obtain an estimate of the modular constant for the functions in this decomposition, both in l^2- and l^∞-norm (Corollary 6.3). An application of Theorem 6.2 is that $\delta\mathcal{E}_p(\Omega)$ is not separable (Corollary 6.5).

The needed facts and notations in the theory of ordered vector space theory compared to pluripotential theory are diverse. Therefore, we give fundamental preliminaries in Section 2, and in Section 3. For further information on the related vector space theory see e.g. [4,32,38], and for more information about pluripotential theory see e.g. [17,25,27].

2. Plurisubharmonic functions

Throughout this article we shall always assume that Ω is a *bounded hyperconvex domain*. A set $\Omega \subseteq \mathbb{C}^n$, $n \geqslant 1$, is a bounded hyperconvex domain if it is a bounded, connected, and open set such that there exists a bounded plurisubharmonic function $\varphi : \Omega \to (-\infty, 0)$ such that the closure of the set $\{z \in \Omega\colon\ \varphi(z) < c\}$ is compact in Ω, for every $c \in (-\infty, 0)$.

We say that a plurisubharmonic function φ defined on Ω belongs to $\mathcal{E}_0$ $(= \mathcal{E}_0(\Omega))$ if $\lim_{z\to\xi}\varphi(z) = 0$, for every $\xi \in \partial\Omega$, and $\int_\Omega (dd^c\varphi)^n < +\infty$, where $(dd^c\cdot)^n$ is the complex Monge–Ampère operator.

Assume that u is a plurisubharmonic function defined on Ω and $\{\varphi_j\}_{j=1}^\infty$, $\varphi_j \in \mathcal{E}_0$, is a decreasing sequence that converges pointwise to u on Ω, as j tends to $+\infty$. If there can be no misinterpretation a sequence $\{\cdot\}_{j=1}^\infty$ will be denoted by $\{\cdot\}$. For $p > 0$ fix, consider the following assertions:

(1) $\sup_j \int_\Omega (-\varphi_j)^p (dd^c\varphi_j)^n < +\infty$,
(2) $\sup_j \int_\Omega (dd^c\varphi_j)^n < +\infty$,

where $(dd^c\cdot)^n$ is the complex Monge–Ampère operator. If the sequence $\{\varphi_j\}$ can be chosen such that (1) holds, then we say that u belongs to $\mathcal{E}_p$, and if (2) holds, then u belongs to $\mathcal{F}$. Let $\mathcal{E}$ $(= \mathcal{E}(\Omega))$ be the class of plurisubharmonic functions φ defined on Ω, such that for each $z_0 \in \Omega$ there exists a neighborhood ω of z_0 in Ω and a decreasing sequence $\{\varphi_j\}$, $\varphi_j \in \mathcal{E}_0$, which converges pointwise to φ on ω and (2) holds. It was proved in [13] that $(dd^c\cdot)^n$ is well defined on $\mathcal{E}$, in the sense that $(dd^c u)^n$ is a non-negative Radon measure for every $u \in \mathcal{E}$. Furthermore, we have that $\mathcal{E}_p, \mathcal{F} \subseteq \mathcal{E}$. To simply the notion we set

$$e_p(u) = \int_\Omega (-u)^p \left(dd^c u\right)^n, \quad p > 0. \tag{2.1}$$

The integral $e_p(u)$ is called the *pluricomplex p-energy* of the function u. A simple, but useful, observations is that if $u, v \in \mathcal{E}_p$, then

$$+\infty > e_p(u+v) \geqslant e_p(u) + e_p(v).$$

With this notation (1.1) can be written as

$$\|u\|_p = \inf_{\substack{u_1-u_2=u\\ u_1,u_2\in\mathcal{E}_p}} e_p(u_1+u_2)^{\frac{1}{n+p}}.$$

Along with $\mathcal{E}_p$ we are also interested in the following set of measures

$$\mathcal{M}_p = \left\{\mu\colon\ \mu \text{ is a positive Radon measure on } \Omega \text{ such that } \left(dd^c u\right)^n = \mu \text{ for some } u \in \mathcal{E}_p\right\}.$$

We shall frequently use the following two theorems from [12] (see also [3]).

Theorem 2.1. *Let $p > 0$. Then the following conditions are equivalent*:

(a) $\mu \in \mathcal{M}_p$,
(b) $\mathcal{E}_p \subset L^p(\mu)$, *and*
(c) *there exists a constant $A \geqslant 0$ such that*

$$\int_\Omega (-u)^p \, d\mu \leqslant A\|u\|_p^p \quad \textit{for all } u \in \mathcal{E}_0.$$

Theorem 2.2 *(The comparison principle). Let $u, v \in \mathcal{E}_p$. If $(dd^c v)^n \leqslant (dd^c u)^n$, then it holds that $u \leqslant v$.*

3. Ordered vector spaces

We say that $\mathcal{K}$ is a *cone* in a vector space X over $\mathbb{R}$ if it is a non-empty subset of X that satisfies:

(1) $\mathcal{K} + \mathcal{K} \subseteq \mathcal{K}$,
(2) $\alpha\mathcal{K} \subseteq \mathcal{K}$ for all $\alpha \geqslant 0$, and
(3) $\mathcal{K} \cap (-\mathcal{K}) = \{0\}$.

Some authors say that $\mathcal{K}$ is a cone if it satisfies (1), and a convex cone if it satisfies (1) and (2). Furthermore, there are literature that say that $\mathcal{K}$ is a pointed convex cone with vertex at zero if it satisfies (1)–(3). We shall not adapted these more detailed notations, since we are here only interested in sets $\mathcal{K}$ with properties (1)–(3). All the sets $\mathcal{E}_p, \mathcal{F}$ and $\mathcal{E}$ are cones (see [12,13]). Theorem 2.1 implies that $\mathcal{M}_p$ is a cone.

An ordered vector space $(L, \succcurlyeq_L)$ is a vector space L with a vector ordering $\succcurlyeq_L$. Let $L^+ = L_+ = \{x \in L\colon\ x \succcurlyeq_L 0\}$ be the positive cone of L. On the other hand, any cone $\mathcal{K}$ in a vector space X generates a vector ordering $\succcurlyeq_X$ defined on X by letting $x \succcurlyeq_X y$ whenever $x - y \in \mathcal{K}$. Hence, $X^+ = \mathcal{K}$.

Now let $\mathcal{K} = \mathcal{E}_p$. As already noted in the introduction we have that $\delta\mathcal{E}_p$ is a vector space over $\mathbb{R}$ with pointwise addition, and usual scalar multiplication. Recall that in the introduction we made the convention that $-\infty - (-\infty) = -\infty$, to handle the case $-\infty - (-\infty)$. We then equip $\delta\mathcal{E}_p$ with the vector ordering induced by $\mathcal{E}_p$, i.e. for $u, v \in \delta\mathcal{E}_p$ we say that $u \succcurlyeq_{\delta\mathcal{E}_p} v$ if $u - v \in \mathcal{E}_p$. It is very important to keep in mind that $u \succcurlyeq_{\delta\mathcal{E}_p} 0$ for all $u \in \mathcal{E}_p$, even though $u(x) \leqslant 0$ for every $x \in \Omega$. One of the major advantages of this construction is that $(\delta\mathcal{E}_p)^+ = \mathcal{E}_p$, and therefore it holds that $\delta\mathcal{E}_p = (\delta\mathcal{E}_p)^+ - (\delta\mathcal{E}_p)^+$. If there is no risk for misunderstanding, then we shall use $\succcurlyeq$ instead of $\succcurlyeq_{\delta\mathcal{E}_p}$. In function theory the classical pointwise ordering $\geqslant$ is defined as $u \geqslant v$ if $u(x) \geqslant v(x)$ for every $x \in \Omega$. The two order relations $\succcurlyeq$ and $\geqslant$ on $\delta\mathcal{E}_p$ are related as follows: if $u \succcurlyeq v$, then $u - v \in \mathcal{E}_p$. Hence, $u \leqslant v$. But as Example 3.1 shows there are functions u, v in $(\delta\mathcal{E}_p, \geqslant)$ with $u \geqslant v$, but u and v are not comparable with respect to $\succcurlyeq$.

Example 3.1. Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$. Let g_w be the pluricomplex Green function with a pole at $w \in \Omega$, and define $u_a = \max(g_w, a)$. Then for $a < b < 0$ we have that $u_a, u_b \in \mathcal{E}_0$, and $u_b \geqslant u_a$. But u_b and u_a are not comparable with respect to $\succcurlyeq$.

Theorem 3.2. *Let $p > 0$. Then we have that*

(a) $(\delta\mathcal{E}_p, \succcurlyeq)$ *is Dedekind σ-complete, i.e. every increasing sequence bounded from above has a supremum*; *and*
(b) $(\delta\mathcal{M}_p, \succcurlyeq)$ *is conditionally complete, i.e. every upper-bounded subset of $(\delta\mathcal{M}_p, \succcurlyeq)$ has a supremum. Furthermore, $(\delta\mathcal{M}_p, \succcurlyeq)$ has the countability property in the sense that for any upper bounded subset $\{\mu_\alpha\}_{\alpha\in I}$ of $(\delta\mathcal{M}_p, \succcurlyeq)$ there exists a countable subset $\{\mu_{\alpha_k}\}$ of $\{\mu_\alpha\}$ such that*

$$\sup_\alpha(\mu_\alpha) = \sup_{k\in\mathbb{N}}(\mu_{\alpha_k}).$$

In particular, $(\delta\mathcal{M}_p, \succcurlyeq)$ is Dedekind σ-complete.

Proof. (a) Assume that $\{u_j\}$ is an increasing sequence in $(\delta\mathcal{E}_p, \succcurlyeq)$, which is upper bounded by ϕ, $\phi \succcurlyeq u_j$ for all $j \in \mathbb{N}$. For each $j \in \mathbb{N}$ set $u_{j+1} - u_j = \alpha_j \in \mathcal{E}_p$. Then we have that

$$u_k = (u_k - u_{k-1}) + (u_{k-1} - u_{k-2}) + \cdots + (u_2 - u_1) + u_1 = u_1 + \sum_{j=1}^{k-1} \alpha_j.$$

Similarly, set $\phi - u_j = \beta_j \in \mathcal{E}_p$. Then

$$\beta_k = \phi - u_k = \phi - u_1 - \sum_{j=1}^{k-1} \alpha_j = \beta_1 - \sum_{j=1}^{k-1} \alpha_j$$

implies that

$$\sum_{j=1}^{k-1} \alpha_j \geqslant \sum_{j=1}^{k-1} \alpha_j + \beta_k = \beta_1.$$

Let $k \to +\infty$, then we get that

$$\sum_{j=1}^{+\infty} \alpha_j \geqslant \beta_1.$$

Hence, the function $\alpha = \sum_{j=1}^{+\infty} \alpha_j$ is a well-defined plurisubharmonic function, as a decreasing limit of plurisubharmonic function. Furthermore, $\alpha \in \mathcal{E}_p$. Let the function v be defined by

$$v = \alpha + u_1 = u_1 + \sum_{j=1}^{+\infty} \alpha_j.$$

To complete this part of the proof we shall now show that $v = \sup_j\{u_j\}$. Note that

$$v - u_k = \alpha + u_1 - \sum_{j=1}^{k-1} \alpha_j - u_1 = \left(\sum_{j=k}^{+\infty} \alpha_j\right) \in \mathcal{E}_p.$$

Thus, $v \succcurlyeq u_k$ for all $k \in \mathbb{N}$. For any upper bound $\psi \in \mathcal{E}_p$ of the sequence $\{u_j\}$ set $\psi - u_j = \gamma_j \in \mathcal{E}_p$, for $j \in \mathbb{N}$. Then we have that

$$\gamma_{k+1} - \gamma_k = \psi - u_{k+1} - (\psi - u_k) = u_k - u_{k+1} = -\alpha_k \geqslant 0,$$

which means that $\{\gamma_k\}$ is a pointwise increasing sequence of plurisubharmonic functions. Furthermore,

$$\gamma_k = (\gamma_k - \gamma_{k-1}) + (\gamma_{k-1} - \gamma_{k-2}) + \cdots + (\gamma_2 - \gamma_1) + \gamma_1 = \gamma_1 - \sum_{j=1}^{k-1} \alpha_j.$$

Hence, the following limit exists:

$$\gamma = \lim_{k\to+\infty} \gamma_k = \gamma_1 - \sum_{j=1}^{+\infty} \alpha_j = \gamma_1 - \alpha,$$

and therefore it follows that $\gamma^* = \gamma_1 - \alpha \geqslant \gamma_1$, where $(w)^*$ denotes the upper semicontinuous regularization of w. Then it follows from [12] that $\gamma^* \in \mathcal{E}_p$. Thus, $\psi - v = \gamma^* \in \mathcal{E}_p$, i.e. $\psi \succcurlyeq v$. We have now proved that $v = \sup\{u_j\}$, which completes part (a) of this proof.

(b) First note that the classical order and the order induced by the cone $\mathcal{M}_p$ coincide. Take $\mu, \nu \in \mathcal{M}_p$. Suppose that $\mu \succcurlyeq \nu$, then $\mu - \nu \in \mathcal{M}_p$ so in particular $\mu \geqslant \nu$. Now suppose that $\mu \geqslant \nu$, then $\mu \geqslant \gamma = \mu - \nu \geqslant 0$ which implies that $\gamma \in \mathcal{M}_p$, so $\mu \succcurlyeq \nu$. Therefore $(\delta\mathcal{M}_p, \succcurlyeq)$ is a Riesz space, since for arbitrary $\mu = \mu_1 - \mu_2, \nu = \nu_1 - \nu_2 \in \delta\mathcal{M}_p$ there exist their supremum and infimum defined as follows

$$\sup(\mu, \nu) = \sup(\mu_1 - \mu_2, \nu_1 - \nu_2) = \sup(\mu_1 + \nu_2, \mu_2 + \nu_1) - (\mu_2 + \nu_2),$$
$$\inf(\mu, \nu) = \inf(\mu_1 - \mu_2, \nu_1 - \nu_2) = \inf(\mu_1 + \nu_2, \mu_2 + \nu_1) - (\mu_2 + \nu_2), \tag{3.1}$$

where

$$\sup(\alpha,\beta)(A)=\sup\bigl\{\alpha(B)+\beta(A\setminus B):\ B\subset A\bigr\},$$
$$\inf(\alpha,\beta)(A)=\inf\bigl\{\alpha(B)+\beta(A\setminus B):\ B\subset A\bigr\},$$

for positive measures α,β. Let $\mathcal{M}_\sigma$ be the family of all σ-finite positive measures defined on Ω. Then $\delta\mathcal{M}_p\subset\delta\mathcal{M}_\sigma$. The space $(\delta\mathcal{M}_\sigma,\geqslant)$ is conditionally complete and then the supremum μ of $\{\mu_\alpha\}_{\alpha\in I}\subset(\mathcal{M}_\sigma,\geqslant)$ is given by

$$\mu(A)=\bigl(\sup(\mu_\alpha)\bigr)(A)=\sup_\alpha\mu_\alpha(A).$$

In general, if a family $\{\mu_\alpha\}_{\alpha\in I}$ is bounded from above by ν, then

$$\sup(\mu_\alpha)=\nu-\inf(\nu-\mu_\alpha).$$

For the lattice $(\delta\mathcal{M}_p,\geqslant)$ it is sufficient to observe that if the upper bound belongs to $(\delta\mathcal{M}_p,\geqslant)$, then supremum is also in $(\delta\mathcal{M}_p,\geqslant)$. Furthermore, $(\delta\mathcal{M}_\sigma,\geqslant)$ has the countability property and therefore also $(\delta\mathcal{M}_p,\geqslant)$. In other words, for any upper bounded subset $\{\mu_\alpha\}_{\alpha\in I}\subset(\delta\mathcal{M}_p,\geqslant)$ there exists a countable subset $\{\mu_{\alpha_k}\}\subseteq\{\mu_\alpha\}$ such that

$$\sup_\alpha(\mu_\alpha)=\sup_{k\in\mathbb{N}}(\mu_{\alpha_k}).\qquad\square$$

Remark. As noted in the proof of Theorem 3.2, the order $\succcurlyeq$ induced by the cone $\mathcal{M}_p$ in $\delta\mathcal{M}_p$ is equivalent to the classical order $\geqslant$. Recall that the classical ordering $\geqslant$ in the measure space $\mathcal{M}_p$ is defined by: if $\mu,\nu\in\delta\mathcal{M}_p$, then we say that $\mu\geqslant\nu$ if $\mu(A)\geqslant\nu(A)$ for every measurable subset $A\subseteq\Omega$. Furthermore, $(\delta\mathcal{M}_p,\succcurlyeq)$ is a Riesz space with the supremum and infimum defined by (3.1).

Remark. Theorem 3.2(a) is also true for $(\delta\mathcal{E}_0,\succcurlyeq)$, $(\delta\mathcal{F},\succcurlyeq)$ and $(\delta\mathcal{E},\succcurlyeq)$. Furthermore, Theorem 3.2(b) is a simple consequence of the well-known results.

Remark. A cone L_+ in an ordered vector space $(L,\succcurlyeq)$ is called *Archimedean* if $y\in L$, $x\in L^+$, and $ny\preccurlyeq x$, for all $n\in\mathbb{N}$, imply that $y\preccurlyeq 0$. Every cone in an ordered, Dedekind σ-complete, vector space is Archimedean. Hence, by Theorem 3.2 $(\delta\mathcal{E}_p,\succcurlyeq)$ and $(\delta\mathcal{M}_p,\succcurlyeq)$ are Archimedean.

4. Quasi-Banach spaces

On several occasions we will use the following theorem. For $p\geqslant 1$, Theorem 4.1 was proved in [33] (see also [12,15]), and for $0<p<1$ in [3]. If $p=0$, then (4.1) can be interpreted as Corollary 5.6 in [13]. Here we give a different proof for the case $0<p<1$, which yields a slightly better constant $D(n,p)$. The difference is that it is left continuous, i.e. $D(n,p)$ converges to 1, as p tends 1^- (for $n\geqslant 2$).

Theorem 4.1. *Let $p>0$ and $u_0,u_1,\dots,u_n\in\mathcal{E}_p$. If $n\geqslant 2$, then*

$$\int_\Omega(-u_0)^p dd^cu_1\wedge\cdots\wedge dd^cu_n\leqslant D(n,p)e_p(u_0)^{p/(p+n)}e_p(u_1)^{1/(p+n)}\cdots e_p(u_n)^{1/(p+n)},\tag{4.1}$$

where

$$D(n,p)=\begin{cases}(\frac{1}{p})^{\frac{n}{n-p}} & \text{if } 0<p<1,\\ 1 & \text{if } p=1,\\ p^{\frac{p\alpha(n,p)}{p-1}} & \text{if } p>1\end{cases}$$

and $\alpha(n,p)=(p+2)(\frac{p+1}{p})^{n-1}-(p+1)$. If $n=1$, then we follow [33] *and interpret* (4.1) *as*

$$\int_\Omega(-u)^p\Delta v\leqslant D(1,p)\biggl(\int_\Omega(-u)^p\Delta u\biggr)^{\frac{p}{p+1}}\biggl(\int_\Omega(-v)^p\Delta v\biggr)^{\frac{1}{p+1}}.$$

Proof. By using standard approximation arguments it is possible to assume, without loss of generality, that $u_0,u_1,\dots,u_n\in\mathcal{E}_0$. Assume also that $0<p<1$. Then $-(-u)^p\in\mathcal{E}_0$. From inequality (2.2) in [3], and Theorem 5.5 in [13] it follows that

$$\int_\Omega (-u)^p \left(dd^c v\right)^n \leqslant \frac{1}{p}\left[\int_\Omega (-v)^p dd^c u \wedge \left(dd^c v\right)^{n-1}\right]^p \left[\int_\Omega (-v)^p \left(dd^c v\right)^n\right]^{1-p}$$
$$\leqslant \frac{1}{p}\left[\int_\Omega (-v)^p \left(dd^c u\right)^n\right]^{\frac{p}{n}} \left[\int_\Omega (-v)^p \left(dd^c v\right)^n\right]^{\frac{n-p}{n}}. \tag{4.2}$$

Similarly we get

$$\int_\Omega (-v)^p \left(dd^c u\right)^n \leqslant \frac{1}{p}\left[\int_\Omega (-u)^p \left(dd^c v\right)^n\right]^{\frac{p}{n}} \left[\int_\Omega (-u)^p \left(dd^c u\right)^n\right]^{\frac{n-p}{n}}. \tag{4.3}$$

By combing (4.2) and (4.3) we get that

$$\int_\Omega (-u)^p \left(dd^c v\right)^n \leqslant \frac{1}{p}\left(\frac{1}{p}\right)^{\frac{p}{n}} \left[\int_\Omega (-u)^p \left(dd^c v\right)^n\right]^{\frac{p^2}{n^2}} e_p(u)^{\frac{p(n-p)}{n^2}} e_p(v)^{\frac{n-p}{n}},$$

hence

$$\int_\Omega (-u)^p \left(dd^c v\right)^n \leqslant \left(\frac{1}{p}\right)^{\frac{n}{n-p}} e_p(u)^{\frac{p}{n+p}} e_p(v)^{\frac{n}{n+p}}.$$

Using this inequality together with Theorem 5.5 in [13] we conclude that

$$\int_\Omega (-u)^p dd^c v_1 \wedge \cdots \wedge dd^c v_n \leqslant \left(\frac{1}{p}\right)^{\frac{n}{n-p}} e_p(u)^{\frac{p}{n+p}} e_p(v_1)^{\frac{1}{n+p}} \cdots e_p(v_n)^{\frac{1}{n+p}},$$

hence $D(n,p) = (1/p)^{\frac{n}{n-p}}$. $\square$

Example 4.2 shows that there are functions u, v, such that for every $p > 2$ the constant $D(2,p)$, in (4.1), is strictly greater than 1.

Example 4.2. Let $\Omega \subset \mathbb{C}^n$, and for $\alpha > 0$ set

$$u_\alpha(z) = |z|^{2\alpha} - 1.$$

Then we have that

$$\int_\Omega (-u_\alpha)^p \left(dd^c u_\beta\right)^n = n(4\pi)^n \frac{\beta^{n+1}}{\alpha} B\left(p+1, \frac{\beta}{\alpha} n\right),$$
$$\int_\Omega (-u_\alpha)^p \left(dd^c u_\alpha\right)^n = n(4\pi)^n \alpha^n B(p+1, n),$$

where $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\, dt$ is the usual beta function. If we assume that $D(n,p) = 1$ in Theorem 4.1, then it holds that

$$\left(\frac{\beta}{\alpha}\right)^{\frac{n+p+np}{n+p}} B\left(p+1, \frac{\beta}{\alpha} n\right) \leqslant B(p+1, n).$$

For $\alpha = 2\beta$ and $n = 2$ we get that

$$2^{-\frac{3p+2}{p+2}} \leqslant \frac{1}{p+2},$$

which is not valid for $p > 2$.

Lemma 4.3. *For all $u, v \in \mathcal{E}_p$, $p > 0$, it holds that*

$$e_p(u+v)^{\frac{1}{n+p}} \leqslant C(n,p)\left(e_p(u)^{\frac{1}{n+p}} + e_p(v)^{\frac{1}{n+p}}\right),$$

where $C(n,p) > 1$ is a constant depending only on n and $p \neq 1$. Furthermore, $C(n,1) = 1$.

Proof. Theorem 4.1 implies that

$$\begin{aligned} e_p(u+v) &= \int_\Omega (-u-v)^p \big(dd^c(u+v)\big)^n \\ &= \sum_{k=0}^n \binom{n}{k} \int_\Omega (-u-v)^p \big(dd^c u\big)^k \wedge \big(dd^c v\big)^{n-k} \\ &\leqslant D(n,p) \sum_{k=0}^n \binom{n}{k} e_p(u+v)^{\frac{p}{n+p}} e_p(u)^{\frac{k}{n+p}} e_p(v)^{\frac{n-k}{n+p}} \\ &= D(n,p) e_p(u+v)^{\frac{p}{n+p}} \big(e_p(u)^{\frac{1}{n+p}} + e_p(v)^{\frac{1}{n+p}}\big)^n, \end{aligned} \tag{4.4}$$

which yields that

$$e_p(u+v) \leqslant D(n,p)^{\frac{n+p}{n}} \big(e_p(u)^{\frac{1}{n+p}} + e_p(v)^{\frac{1}{n+p}}\big)^{n+p},$$

and

$$e_p(u+v)^{\frac{1}{n+p}} \leqslant D(n,p)^{\frac{1}{n}} \big(e_p(u)^{\frac{1}{n+p}} + e_p(v)^{\frac{1}{n+p}}\big).$$

Thus, $C(n,p) = D(n,p)^{\frac{1}{n}}$. $\square$

Lemma 4.4. *Let $u \in \mathcal{E}_p$, $p > 0$. Then*

$$\|u\|_p = e_p(u)^{\frac{1}{n+p}}.$$

Proof. If $u = u - 0$ in the definition of the quasi-norm, then we get that

$$\|u\|_p \leqslant e_p(u)^{\frac{1}{n+p}}.$$

Let $u_1, u_2 \in \mathcal{E}_p$ be such that $u = u_1 - u_2$, then $u \geqslant u + 2u_2 = u_1 + u_2$ and

$$e_p(u)^{\frac{1}{n+p}} \leqslant \left(\int_\Omega (-u_1-u_2)^p \big(dd^c(u+2u_2)\big)^n\right)^{\frac{1}{n+p}} = e_p(u_1+u_2)^{\frac{1}{n+p}}.$$

Taking infimum over u_1, u_2, $u_1 - u_2 = u$, yields that

$$e_p(u)^{\frac{1}{n+p}} \leqslant \|u\|_p. \qquad \square$$

Lemma 4.5. *Let $p > 0$, and let $\|\cdot\|_p$ be defined on $\delta\mathcal{M}_p$ by*

$$|\mu|_p = \inf_{\substack{\mu_1-\mu_2=\mu \\ \mu_1,\mu_2\in\mathcal{M}_p}} \|u_{\mu_1}\|_p^n + \|u_{\mu_2}\|_p^n,$$

where $u_{\mu_j} \in \mathcal{E}_p$, $j = 1, 2$, are the uniquely determined solutions to the equations $(dd^c u_{\mu_j})^n = \mu_j$. Then,

$$|\mu|_p = \|u_{\mu^+}\|_p^n + \|u_{\mu^-}\|_p^n,$$

where $\mu^+ = \frac{1}{2}(|\mu| + \mu)$ and $\mu^- = \frac{1}{2}(|\mu| - \mu)$. Furthermore, we have that if $\mu \in \mathcal{M}_p$, then $|\mu|_p = \|u_\mu\|_p^n$.

Proof. The decomposition $\mu = \mu^+ - \mu^-$ has the minimal property in the sense that for any decomposition $\mu = \mu_1 - \mu_2$ we have $\mu^+ \leqslant \mu_1$ and $\mu^- \leqslant \mu_2$. Hence,

$$\big(dd^c u_{\mu_1}\big)^n \geqslant \big(dd^c u_{\mu^+}\big)^n,$$

and by Theorem 2.2 we have that $u_{\mu_1} \leqslant u_{\mu^+}$. Thus,

$$\|u_{\mu_1}\|_p^n = \left(\int_\Omega (-u_{\mu_1})^p \, d\mu_1\right)^{\frac{n}{n+p}} \geqslant \left(\int_\Omega (-u_{\mu^+})^p \, d\mu^+\right)^{\frac{n}{n+p}} = \|u_{\mu^+}\|_p^n.$$

Similarly, we get that $\|u_{\mu_2}\|_p^n \geqslant \|u_{\mu^-}\|_p^n$, and this implies that

$$|\mu|_p = \|u_{\mu^+}\|_p^n + \|u_{\mu^-}\|_p^n.$$

The last statement of this lemma follows immediately, since for $\mu \in \mathcal{M}_p$ we have $\mu^+ = \mu$ and $\mu^- = 0$. □

Let us recall the definition of a quasi-Banach space.

Definition 4.6. A *quasi-norm* $\|\cdot\|$ on a vector space X is a mapping $\|\cdot\| : X \to [0, +\infty)$ with the following properties:

(1) $\|x\| = 0$ if and only if $x = 0$,
(2) $\|tx\| = |t|\|x\|$ for all $x \in X$ and $t \in \mathbb{R}$,
(3) there exists a constant $C \geqslant 1$ such that for all $x, y \in X$ we have that

$$\|x + y\| \leqslant C\bigl(\|x\| + \|y\|\bigr).$$

The constant C is often refereed to the modulus of concavity of the quasi-norm $\|\cdot\|$. The classical Aoki–Rolowicz theorem for quasi-Banach spaces [6,35] states that every quasi-normed space X is q-normable for some $0 < q \leqslant 1$. In other words, X can be endowed with an equivalent quasi-norm $|||\cdot|||$ that is q-subadditive, and therefore we can define the following metric $d(u, v) = |||u - v|||^q$ on X. The vector space X is called a *quasi-Banach space* if it is complete with respect to the metric d induced by the quasi-norm $\|\cdot\|$.

Theorem 4.7. *Let $p > 0$.*

(a) *If $\|\cdot\|_p$ is defined by* (1.1)*, then $(\delta\mathcal{E}_p, \|\cdot\|_p)$ is a quasi-Banach space for $p \neq 1$, and $(\delta\mathcal{E}_1, \|\cdot\|_1)$ is a Banach space.*
(b) *If $|\cdot|_p$ is defined as in Lemma* 4.5*, then $(\delta\mathcal{M}_p, |\cdot|_p)$ is a quasi-Banach space for $p \neq 1$, and $(\delta\mathcal{M}_1, |\cdot|_1)$ is a Banach space.*

Proof. In this proof we shall use an idea from [16].

Part (a):

(1) We start by proving that $\|u\|_p = 0$ if and only if $u = 0$. If $u = 0$, then it is clear that $\|u\|_p = 0$. Assume that $\|u\|_p = 0$, and let $\epsilon > 0$. Then there exist functions $u_1, u_2 \in \mathcal{E}_p$ such that $u_1 - u_2 = u$, and

$$e_p(u_1 + u_2) < \epsilon.$$

Furthermore, since $u_1 + u_2 \in \mathcal{E}_p$ there exists by Lemma 2.1 in [14] a pointwise decreasing sequence $\{v_j\}$, $v_j \in \mathcal{E}_0$, that converges pointwise to $u_1 + u_2$, and

$$\sup_j e_p(v_j) \leqslant e_p(u_1 + u_2) < \epsilon.$$

Let $\phi \in \mathcal{E}_0$ be such that $(dd^c\phi)^n = d\lambda_n$, where λ_n is the Lebesgue measure (see e.g. [1]). From Theorem 4.1 it follows that

$$\|v_j\|_{L^p}^p = \int_\Omega (-v_j)^p \, d\lambda_n = \int_\Omega (-v_j)^p \bigl(dd^c\phi\bigr)^n \leqslant D(n, p) e_p(\phi)^{\frac{n}{n+p}} e_p(v_j)^{\frac{p}{n+p}} \leqslant C\epsilon^{\frac{p}{n+p}},$$

where $C \geqslant 0$ is a constant independent of j. Hence, $\|u\|_{L^p}^p \leqslant \|u_1 + u_2\|_{L^p}^p \leqslant C\epsilon^{\frac{p}{n+p}}$, since $|u| = |u_1 - u_2| \leqslant -u_1 - u_2$. If $\epsilon \to 0^+$, then $\|u\|_{L^p} = 0$, and therefore we get that $u = 0$ almost everywhere w.r.t. $d\lambda_n$. The function u is plurisubharmonic, hence $u = 0$ everywhere on Ω.

(2) In this step we prove homogeneity, i.e. for all $t \in \mathbb{R}$, and all $u \in \delta\mathcal{E}_p$ we have that

$$\|tu\|_p = |t|\|u\|_p.$$

The definition of $\|\cdot\|_p$ implies that

$$\begin{aligned}\|tu\|_p &= \inf\bigl\{e_p(u_1 + u_2)^{1/(n+p)}\colon tu = u_1 - u_2,\ u_1, u_2 \in \mathcal{E}_p\bigr\}\\ &= \inf\bigl\{e_p(tv_1 + tv_2)^{1/(n+p)}\colon u = v_1 - v_2,\ v_1, v_2 \in \mathcal{E}_p\bigr\} = |t|\|u\|_p.\end{aligned}$$

(3) To finish the proof that $\|\cdot\|$ is a quasi-norm, we shall show that there exists a constant $C \geqslant 1$ such that for all $u, v \in \delta\mathcal{E}_p$ we have that

$$\|u + v\|_p \leqslant C\bigl(\|u\|_p + \|v\|_p\bigr). \tag{4.5}$$

Let $\epsilon > 0$. Then there exist functions $u_1, u_2, v_1, v_2 \in \mathcal{E}_p$ such that $u_1 - u_2 = u$, $v_1 - v_2 = v$,

$$e_p(u_1 + u_2)^{\frac{1}{n+p}} < \|u\|_p + \epsilon \quad \text{and} \quad e_p(v_1 + v_2)^{\frac{1}{n+p}} < \|v\|_p + \epsilon.$$

Hence,

$$\|u + v\|_p \leqslant e_p(u_1 + u_2 + v_1 + v_2)^{\frac{1}{n+p}} \leqslant C\big(e_p(u_1 + u_2)^{\frac{1}{n+p}} + e_p(v_1 + v_2)^{\frac{1}{n+p}}\big) \leqslant C\big(\|u\|_p + \|v\|_p\big) + 2C\epsilon,$$

and when we let $\epsilon \to 0^+$ inequality (4.5) is obtained. Note that if $p = 1$, then we can take $C = 1$. In other words, if $p = 1$, then $\|\cdot\|_1$ is not only a quasi-norm but also a norm.

(4) Assume that $\{u_j\}$ is a Cauchy sequence in $(\delta\mathcal{E}_p, \|\cdot\|_p)$. Then there exists an increasing subsequence $\{j_k\}$ such that for each $k \in \mathbb{N}$ it holds that

$$\|u_{j_{k+1}} - u_{j_k}\|_p \leqslant (2C)^{-k},$$

where C is the modulus of concavity of the quasi-norm $\|\cdot\|_p$. There also exist functions $\varphi_k^1, \varphi_k^2 \in \mathcal{E}_p$ such that

$$u_{j_{k+1}} - u_{j_k} = \varphi_k^1 - \varphi_k^2,$$

and

$$e_p\big(\varphi_k^1 + \varphi_k^2\big)^{\frac{1}{n+p}} \leqslant (2C)^{-k-1} + \|u_{j_{k+1}} - u_{j_k}\|_p.$$

Define now the sequences $\{\psi_k^1\}$ and $\{\psi_k^2\}$ by

$$\psi_k^1 = \sum_{j=1}^{k} \varphi_j^1 \quad \text{and} \quad \psi_k^2 = \sum_{j=1}^{k} \varphi_j^2.$$

These definitions yield that $\psi_k^1, \psi_k^2 \in \mathcal{E}_p$, and $\{\psi_k^1\}$ and $\{\psi_k^2\}$ are decreasing. Furthermore, we have that

$$u_{j_{k+1}} = u_{j_1} + \sum_{i=1}^{k} (u_{j_{i+1}} - u_{j_i}).$$

Therefore we have that

$$\begin{aligned}
\big(\max\big(e_p\big(\psi_k^1\big), e_p\big(\psi_k^2\big)\big)\big)^{\frac{1}{n+p}} &\leqslant e_p\big(\psi_k^1 + \psi_k^2\big)^{\frac{1}{n+p}} = e_p\left(\sum_{j=1}^{k} \big(\varphi_j^1 + \varphi_j^2\big)\right)^{\frac{1}{n+p}} \\
&\leqslant \sum_{j=1}^{k} C^j e_p\big(\varphi_j^1 + \varphi_j^2\big)^{\frac{1}{n+p}} \leqslant \sum_{l=1}^{k} C^l\big((2C)^{-l-1} + \|u_{j_{l+1}} - u_{j_l}\|_p\big) \leqslant \widetilde{C},
\end{aligned}$$

where $\widetilde{C}$ is a constant that does not depend on k. Thus, there exist functions $u_1, u_2 \in \mathcal{E}_p$ such that $\psi_k^1 \to u_1$, and $\psi_k^2 \to u_2$, in $(\delta\mathcal{E}_p, \|\cdot\|_p)$, as $k \to +\infty$. Therefore we have that $u_{j_k} \to u$ and $u_1 - u_2 = u \in \delta\mathcal{E}_p$.

Part (b):

(i) Let $\mu \in \delta\mathcal{M}_p$. Then the following assertions are equivalent

- $|\mu|_p = 0$,
- $\|u_{\mu^+}\|_p = \|u_{\mu^-}\|_p = 0$,
- $u_{\mu^+} = u_{\mu^-} = 0$,
- $\mu^+ = \mu^- = 0$,
- $\mu = 0$.

Thus, $|\mu|_p = 0$ if and only if $\mu = 0$.

(ii) For $t \geqslant 0$ we have that

$$(t\mu)^+ = t\mu^+, \qquad (t\mu)^- = t\mu^-, \qquad u_{t\mu^+} = t^{\frac{1}{n}} u_{\mu^+} \quad \text{and} \quad u_{t\mu^-} = t^{\frac{1}{n}} u_{\mu^-}.$$

Therefore we have that

$$|t\mu|_p = \|u_{(t\mu)^+}\|_p^n + \|u_{(t\mu)^-}\|_p^n = \big\|t^{\frac{1}{n}} u_{\mu^+}\big\|_p^n + \big\|t^{\frac{1}{n}} u_{\mu^-}\big\|_p^n = t|\mu|_p.$$

For $t < 0$, we have in a similar manner

$$(t\mu)^+ = (-t)\mu^-, \qquad (t\mu)^- = (-t)\mu^+, \qquad u_{t\mu^+} = |t|^{\frac{1}{n}} u_{\mu^-} \quad \text{and} \quad u_{t\mu^-} = |t|^{\frac{1}{n}} u_{\mu^+}.$$

Hence,

$$|t\mu|_p = |t||\mu|_p.$$

(iii) Let $\mu, \nu \in \delta\mathcal{M}_p$. Then we have that

$$\mu = \mu^+ - \mu^-, \qquad \nu = \nu^+ - \nu^-, \qquad (\mu+\nu)^+ \leqslant \mu^+ + \nu^+ \quad \text{and} \quad (\mu+\nu)^- \leqslant \mu^- + \nu^-.$$

Theorem 2.1 implies that there exists a uniquely determined function $u_{(\mu+\nu)^+} \in \mathcal{E}_p$ such that $(dd^c u_{(\mu+\nu)^+})^n = (\mu+\nu)^+$, therefore we have that

$$\begin{aligned} e_p(u_{(\mu+\nu)^+}) &= \int_\Omega (-u_{(\mu+\nu)^+})^p \big(dd^c u_{(\mu+\nu)^+}\big)^n = \int_\Omega (-u_{(\mu+\nu)^+})^p (\mu+\nu)^+ \\ &\leqslant \int_\Omega (-u_{(\mu+\nu)^+})^p \big(\mu^+ + \nu^+\big) = \int_\Omega (-u_{(\mu+\nu)^+})^p \big(\big(dd^c u_{\mu^+}\big)^n + \big(dd^c u_{\nu^+}\big)^n\big) \\ &\leqslant D(n,p) e_p(u_{(\mu+\nu)^+})^{\frac{p}{n+p}} \big(e_p(u_{\mu^+})^{\frac{n}{n+p}} + e_p(u_{\nu^+})^{\frac{n}{n+p}}\big). \end{aligned}$$

Thus,

$$e_p(u_{(\mu+\nu)^+})^{\frac{n}{n+p}} \leqslant D(n,p)\big(e_p(u_{\mu^+})^{\frac{n}{n+p}} + e_p(u_{\nu^+})^{\frac{n}{n+p}}\big).$$

In a similar manner we get that

$$e_p(u_{(\mu+\nu)^-})^{\frac{n}{n+p}} \leqslant D(n,p)\big(e_p(u_{\mu^-})^{\frac{n}{n+p}} + e_p(u_{\nu^-})^{\frac{n}{n+p}}\big).$$

We now have that

$$\begin{aligned} |\mu+\nu|_p &= \|u_{(\mu+\nu)^+}\|_p^n + \|u_{(\mu+\nu)^-}\|_p^n = e_p(u_{(\mu+\nu)^+})^{\frac{n}{n+p}} + e_p(u_{(\mu+\nu)^-})^{\frac{n}{n+p}} \\ &\leqslant D(n,p)\big(e_p(u_{\mu^+})^{\frac{n}{n+p}} + e_p(u_{\nu^+})^{\frac{n}{n+p}} + e_p(u_{\mu^-})^{\frac{n}{n+p}} + e_p(u_{\nu^-})^{\frac{n}{n+p}}\big) \\ &= D(n,p)\big(\|u_{\mu^+}\|_p^n + \|u_{\mu^-}\|_p^n + \|u_{\nu^+}\|_p^n + \|u_{\nu^-}\|_p^n\big) = D(n,p)\big(\|\mu\|_p + \|\nu\|_p\big). \end{aligned}$$

From (i), (ii) and (iii) it now follows that $|\cdot|_p$ is a quasi-norm on $\delta\mathcal{M}_p$. If $p = 1$, then we have that $C = 1$. In other words, $|\cdot|_1$ is a norm.

(iv) In this part we shall prove completeness in $(\delta\mathcal{M}_p, |\cdot|_p)$. Assume that $\{\mu_j\}$ is a Cauchy sequence in $(\delta\mathcal{M}_p, |\cdot|_p)$. Then there exists an increasing subsequence $\{j_k\}$ such that for each $k \in \mathbb{N}$ we have that

$$|\mu_{j_{k+1}} - \mu_{j_k}|_p = \|u_{(\mu_{j_{k+1}} - \mu_{j_k})^+}\|_p^n + \|u_{(\mu_{j_{k+1}} - \mu_{j_k})^-}\|_p^n \leqslant \big(2C(n,p)\big)^{-nk}, \tag{4.6}$$

where $C(n,p)$ is the constant from Lemma 4.3. If we define

$$\mu = \mu_{j_1} + \sum_{k=1}^{+\infty} (\mu_{j_{k+1}} - \mu_{j_k}), \quad \text{then} \quad \mu^+ \leqslant \mu_{j_1}^+ + \sum_{k=1}^{+\infty} (\mu_{j_{k+1}} - \mu_{j_k})^+.$$

The decreasing sequence $\{v_k^+\}$ defined by

$$v_k^+ = \sum_{l=1}^{k} u_{(\mu_{j_{l+1}} - \mu_{j_l})^+},$$

satisfies, by using (4.6),

$$\begin{aligned} \sup_k e_p(v_k) &= \sup_k e_p\left(\sum_{l=1}^{k} u_{(\mu_{j_{l+1}} - \mu_{j_l})^+}\right) \\ &\leqslant \sup_k \left(\sum_{l=1}^{k} C(n,p)^l e_p(u_{(\mu_{j_{l+1}} - \mu_{j_l})^+})^{\frac{1}{n+p}}\right)^{n+p} \\ &\leqslant \sup_k \left(\sum_{l=1}^{k} C(n,p)^l \big(2C(n,p)\big)^{-l}\right)^{n+p} < +\infty. \end{aligned}$$

Hence, there exists a function $v^+ \in \mathcal{E}_p$ such that $\{v_k^+\}$ converges pointwise to v^+, as $k \to +\infty$. Furthermore,

$$\mu^+ \leqslant \mu_{j_1}^+ + \sum_{k=1}^{+\infty} (\mu_{j_{k+1}} - \mu_{j_k})^+ \leqslant \left(dd^c\left(u_{j_1} + v^+\right)\right)^n.$$

This means that $\mu^+ \in \mathcal{M}_p$. In a similar manner we get that $\mu^- \in \mathcal{M}_p$. Therefore this proof is completed since $\mu_{j_k} \to \mu$ in $(\delta\mathcal{M}_p, |\cdot|_p)$, and $\mu^+ - \mu^- = \mu \in \delta\mathcal{M}_p$. $\square$

Corollary 4.8. *Let $p > 0$. Then*

(a) *$\mathcal{E}_p$ is closed in $(\delta\mathcal{E}_p, \|\cdot\|_p)$, and*
(b) *$\mathcal{M}_p$ is closed in $(\delta\mathcal{M}_p, |\cdot|_p)$.*

Proof. This follows by a similar proof as for Theorem 4.7. $\square$

Theorem 4.9. *Let $p > 0$. Then the interior of $\mathcal{E}_p$ in $(\delta\mathcal{E}_p, \|\cdot\|_p)$ is empty. The correspondent statement for $(\delta\mathcal{M}_p, |\cdot|_p)$ is also valid.*

Proof. *Case $\mathcal{E}_p$*: The point 0 is not an interior point of $\mathcal{E}_p$, since $\|\cdot\|_p$ is homogeneous. Assume that $0 \neq u \in \mathcal{E}_p$ is an interior point of $(\delta\mathcal{E}_p, \|\cdot\|_p)$. Then there exists $\epsilon > 0$ such that if $\|u - v\|_p < \epsilon$, then $v \in \mathcal{E}_p$. Let $\mu = (dd^c u)^n$. We can find a subset B of Ω such that $\mu(B) > 0$, and

$$2^{\frac{1}{n}} \left(\int_B (-u)^p \, d\mu \right)^{\frac{1}{n+p}} < \epsilon.$$

Let χ_B be the characteristic function for the set B in Ω. Then there exists a function $\phi \in \mathcal{E}_p$ such that $(dd^c\phi)^n = 2\chi_B(dd^c u)^n$. The function defined by $v = u - \phi$ is not in $\mathcal{E}_p$, since $(dd^c u)^n$ is not bigger than $(dd^c\phi)^n$. To obtain a contradiction and in that way complete this proof it is sufficient to prove that $\|u - v\|_p < \epsilon$. We have that

$$\left(dd^c\phi\right)^n = 2\chi_B\left(dd^c u\right)^n \leqslant 2\left(dd^c u\right)^n = \left(dd^c 2^{\frac{1}{n}} u\right)^n.$$

Using Theorem 2.2 we obtain that $2^{\frac{1}{n}} u \leqslant \phi$. Hence,

$$\|u - v\|_p = \|\phi\|_p = e_p(\phi)^{\frac{1}{n+p}} = \left(\int_\Omega (-\phi)^p \left(dd^c\phi\right)^n \right)^{\frac{1}{n+p}} \leqslant \left(2 \int_B \left(-2^{\frac{1}{n}} u\right)^p \left(dd^c u\right)^n \right)^{\frac{1}{n+p}} < \epsilon.$$

Case $\mathcal{M}_p$: This part follows as in the previous part. The point $0 \in \mathcal{M}_p$ is not an interior point of $\mathcal{M}_p$ in $(\delta\mathcal{M}_p, |\cdot|_p)$. Assume that $0 \neq \mu \in \mathcal{M}_p$ is an interior point of $\mathcal{M}_p$, i.e. there exists $\epsilon > 0$ such that if $|\mu - \nu|_p < \epsilon$, then $\nu \in \mathcal{M}_p$. Then there exists $u_\mu \in \mathcal{E}_p$ such that $(dd^c u_\mu)^n = d\mu$ and, as before, there exists $B \subset \Omega$ such that $\mu(B) > 0$, and

$$\left(\int_B (-u_\mu)^p \, d\mu \right)^{\frac{n}{n+p}} < \frac{\epsilon}{2}.$$

The measure $\nu = \chi_{\Omega \setminus B}\mu - \chi_B\mu$ is not an element of $\mathcal{M}_p$, and we have obtained a contradiction when we proved that $|\mu - \nu|_p < \epsilon$. Using the inequality $\chi_B\mu \leqslant \mu$ together with Theorem 2.2 we get that $u_\mu \leqslant u_{\chi_B\mu}$, where $u_{\chi_B\mu} \in \mathcal{E}_p$ is such that $(dd^c u_{\chi_B\mu})^n = \chi_B\mu$. Hence,

$$|\mu - \nu|_p = 2|\chi_B\mu|_p = 2\|u_{\chi_B\mu}\|_p^n = 2\left(\int_\Omega (-u_{\chi_B\mu})^p \left(dd^c u_{\chi_B\mu}\right)^n \right)^{\frac{n}{n+p}} \leqslant 2\left(\int_B (-u_\mu)^p \, d\mu \right)^{\frac{n}{n+p}} < \epsilon. \qquad \square$$

Remark. In a similar manner, one can obtain Theorem 4.9 for the cones $\mathcal{F}$ and $\mathcal{E}$.

5. Duality

We shall start this section by recalling some definitions and introduce some notation. The *algebraic dual* of a vector space X, i.e. the vector space of all linear functionals on X, is denoted by X^*. In a given ordered vector space $(X, \succcurlyeq)$ a linear functional $f : X \to \mathbb{R}$ is called

(a) *positive*, if $f(x) \geqslant 0$ holds for all $x \in X^+$,
(b) *regular*, if f can be written as a difference of two positive operators,

(c) *ordered bounded*, if $f([u,v])$ is bounded for all $u, v \in X$, where the order interval $[u,v]$ is defined as

$$[u,v] = \{w \in X\colon\ v \succcurlyeq w \succcurlyeq u\}.$$

The set of all regular linear functional defined on $(X, \succcurlyeq)$, is denoted by X^r, and the set of ordered bounded linear functional is denoted by X^b. These vector subspaces satisfy $X^r \subseteq X^b \subseteq X^*$.

The *topological dual* of a topological vector space (X,τ) is denoted by X', i.e. X' is the vector subspace of X^* consisting of all τ-continuous functionals. Let $\mathcal{K}$ be any cone in (X,τ), then we define the *dual cone* $\mathcal{K}'$ of $\mathcal{K}$ in (X,τ) by

$$\mathcal{K}' = \bigl\{f \in X^*\colon\ f(x) \geqslant 0 \text{ for each } x \in \mathcal{K}\bigr\}.$$

A cone $\mathcal{K}$ in a topological vector space (X,τ) is called τ*-normal* whenever τ has a base at zero consisting of $\mathcal{K}$ full sets. If there is no ambiguity about the topology, then we simply shall say normal cone.

In the context of normal cones we need the following known results (see e.g. [37]):

- Let $\mathcal{K}$ be a cone in an ordered, topological vector space $(X, \succcurlyeq, \tau)$. If for any two nets $\{x_\alpha\}_{\alpha\in I}$ and $\{y_\alpha\}_{\alpha\in I}$ of $(X, \succcurlyeq, \tau)$ such that $x_\alpha \succcurlyeq y_\alpha \succcurlyeq 0$ for each $\alpha \in I$ the condition $x_\alpha \xrightarrow{\tau} 0$ implies that $y_\alpha \xrightarrow{\tau} 0$, then $\mathcal{K}$ is normal cone.
- A cone $\mathcal{K}$ in a normed space $(X, \|\cdot\|)$ is normal, if there exists a constant $A > 0$ such that $\|u\| \leqslant A\|u+v\|$ for all $u, v \in \mathcal{K}$.

Before arriving to the main theorem of this section we need the following auxiliary results.

Lemma 5.1. *Let $(X, \succcurlyeq, \tau)$ be an ordered topological vector space such that the positive cone X^+ is generating, and τ-closed, such that the linear topology τ is completely metrizable. If $x_n \xrightarrow{\tau} 0$ in $(X, \succcurlyeq, \tau)$, then there exist a subsequence $\{u_n\}$ of $\{x_n\}$ and some $y \in X^+$ such that for each n we have that*

$$\frac{1}{n}y \succcurlyeq u_n \succcurlyeq -\frac{1}{n}y.$$

Furthermore, an operator $T : X \to \mathbb{R}$ is continuous if and only if $T : X^+ \to \mathbb{R}$ is continuous at 0.

Proof. For the first statement see e.g. Lemma 2.30 in [4], and for the second statement see e.g. Corollary 2.31 in [4]. □

Lemma 5.2. *Let $(X, \succcurlyeq, \tau)$ be an ordered topological vector space with the same assumptions as in Lemma* 5.1*. Then $X^b \subseteq X'$, i.e. all ordered bounded functionals on X are continuous.*

Proof. By the second statement in Lemma 5.1 it is sufficient to prove that any $T \in X^b$ is continuous at 0. Let $\{x_n\}$ be a sequence of $(X, \succcurlyeq, \tau)$ with $x_n \xrightarrow{\tau} 0$. Then by using the first statement in Lemma 5.1, we can choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, and some $y \in X^+$ such that for each n_k we have that

$$\frac{1}{n_k}y \succcurlyeq x_{n_k} \succcurlyeq -\frac{1}{n_k}y.$$

Since $T \in X^b$ then, by definition, there exists a constant $C > 0$ such that $|T([-y,y])| \leqslant C$. Therefore we have that

$$\bigl|T(x_{n_k})\bigr| \leqslant \left|T\left(\left[-\frac{1}{n_k}y, \frac{1}{n_k}y\right]\right)\right| \leqslant \frac{1}{n_k}C. \tag{5.1}$$

Hence, $T(x_{n_k}) \to 0$. This yields that T is continuous at 0. To see this assume that T is not continuous at 0, i.e. there exist a sequence $\{z_n\}$ of X that converges to 0, and a constant $A > 0$ such that $|T(z_n)| > A$. But then we can pass to a subsequence $\{z_{n_k}\}$ with the property (5.1), and then $T(z_{n_k}) \to 0$. Hence, a contradiction has been obtained. Thus, T is continuous at 0. □

Remark. Let $(X, \|\cdot\|)$ be a quasi-normed space. Then the topology generated by $\|\cdot\|$ is a linear topology, i.e. $(x,y) \mapsto x+y$ from $X \times X$ to X, and $(\alpha, x) \mapsto \alpha x$ from $\mathbb{R} \times X$ to X, are both continuous mappings. In other words, every quasi-normed space is a topological vector space. Hence, Theorem 4.7 and Corollary 4.8 imply that the assumptions in Lemma 5.2 are satisfied for $(\delta\mathcal{E}_p, \succcurlyeq, \|\cdot\|_p)$ and $(\delta\mathcal{M}_p, \succcurlyeq, \|\cdot\|_p)$.

Let Ω be bounded hyperconvex domain in $\mathbb{C}^n$. For each non-pluripolar set $\omega \Subset \Omega$ we define

$$D_\omega : \mathcal{E}_p \ni u \to \int\limits_\omega \Delta u \in \mathbb{R}.$$

Then D_ω is a positive, and linear, functional defined on $\mathcal{E}_p$. Hence, it can be extended to a regular, linear functional, defined on $(\delta\mathcal{E}_p, \succcurlyeq, \|\cdot\|_p)$. Let $\mathcal{D}$ denote the family of the functionals D_ω together with the zero functional, i.e.

$$\mathcal{D} = \{0\} \cup \{D_\omega\colon\ \omega \Subset \Omega\}.$$

In particular, we have that $\mathcal{D} \subset \mathcal{E}_p'$.

We need the following notion from standard functional theory (see e.g. [22]): Let X be a Banach space, and let $A \subset X'$. Then the set A is said to *separate points of* X if for all $0 \neq x \in X$ there exists $f \in A$ such that $f(x) \neq 0$. A set A separates points of X if and only if the $\sigma(X', X)$-closure of the linear span of A is X'. Here $\sigma(X', X)$ is usual weak*-topology of X'. This is also valid for quasi-Banach spaces X with the property that X' separates points of X.

Lemma 5.3. *For $p \geqslant 1$, we have that*

(1) *$\delta\mathcal{M}_p \subset (\delta\mathcal{E}_p)'$, and $\mathcal{M}_p$ separates points of $(\delta\mathcal{E}_p, \|\cdot\|_p)$,*
(2) *$\delta\mathcal{E}_p \subset (\delta\mathcal{M}_p)'$, and $\mathcal{E}_p$ separates points of $(\delta\mathcal{M}_p, |\cdot|_p)$.*

Furthermore, for $p > 0$,

(3) *the family $\mathcal{D}$ separates points of $(\delta\mathcal{E}_p, \|\cdot\|_p)$.*

Proof. *Part* (1): Let $p \geqslant 1$. We start by constructing a continuous, linear mapping $T : \delta\mathcal{M}_p \ni \mu \to T_\mu \in (\delta\mathcal{E}_p)'$, which is injective. In this way we identify $\delta\mathcal{M}_p$ with a subset of $(\delta\mathcal{E}_p)'$. Fix $w \in \mathcal{E}_0 \cap C^\infty(\Omega)$ such that $e_p(w) = D(n,p)^{\frac{n+p}{1-p}}$ and fix $\mu \in \delta\mathcal{M}_p$ (for $p = 1$ take $w = -1$). Assume that μ is not the zero measure, and let $T_\mu : \delta\mathcal{E}_p \to \mathbb{R}$ be defined by

$$T_\mu(u) = T_\mu(u_1 - u_2) = \int_\Omega (u_1 - u_2)(-w)^{p-1}\, d\mu \in \mathbb{R}.$$

Theorem 5.1 in [12] (see also [3]) yields that there exist a uniquely determined functions $u_{\mu^+}, u_{\mu^-} \in \mathcal{E}_p$ with $(dd^c u_{\mu^+})^n = \mu^+$, $(dd^c u_{\mu^-})^n = \mu^-$ and therefore it follows that

$$\begin{aligned}
\left|T_\mu(u)\right| &= \left|T_\mu(u_1 - u_2)\right| = \left|\int_\Omega (u_1 - u_2)(-w)^{p-1}\big(d\mu^+ - d\mu^-\big)\right| \\
&\leqslant \int_\Omega (-u_1 - u_2)(-w)^{p-1}\Big(\big(dd^c v_{\mu^+}\big)^n + \big(dd^c v_{\mu^-}\big)^n\Big) \\
&\leqslant D(n,p) e_p(u_1 + u_2)^{\frac{1}{n+p}} e_p(w)^{\frac{p-1}{n+p}} \big(e_p(v_{\mu^+})^{\frac{n}{n+p}} + e_p(v_{\mu^-})^{\frac{n}{n+p}}\big) = e_p(u_1 + u_2)^{\frac{1}{n+p}} |\mu|_p.
\end{aligned}$$

Taking infimum over all decomposition of u we get

$$\left|T_\mu(u)\right| \leqslant |\mu|_p \|u\|_p,$$

thus, $T_\mu \in (\delta\mathcal{E}_p)'$ and $\|T_\mu\| \leqslant |\mu|_p$.

We shall next prove that T is injective. Assume that for some $\mu, \nu \in \delta\mathcal{M}_p$ we have $T(\mu) = T(\mu^+ - \mu^-) = T(\nu^+ - \nu^-) = T(\nu)$, i.e. for all $u \in \delta\mathcal{E}_p$ we have that

$$\int_\Omega u(-w)^{p-1}\big(d\mu^+ - d\mu^-\big) = \int_\Omega u(-w)^{p-1}\big(d\nu^+ - d\nu^-\big).$$

The space of infinitely differentiable functions with compact support, C_0^∞, is dense in $\delta\mathcal{E}_p$ [13], and therefore it follows that

$$\int_\Omega \psi\big(d\mu^+ - d\mu^-\big) = \int_\Omega \psi\big(d\nu^+ - d\nu^-\big),$$

for all $\psi \in C_0^\infty$. Thus, $\mu = \nu$. Hence, $\delta\mathcal{M}_p$ is continuously embedded into $(\delta\mathcal{E}_p)'$.

To finish part (1) we shall now prove that $\mathcal{M}_p$ separates points of $\delta\mathcal{E}_p$, i.e. for all $0 \neq x \in \delta\mathcal{E}_p$ there exists $f \in \mathcal{M}_p$ such that $f(x) \neq 0$. Take any $u \in \delta\mathcal{E}_p$, then $u = u_1 - u_2$ for some $u_1, u_2 \in \mathcal{E}_p$, and without loss of generality we can assume that $u_1 \neq u_2$. Consider the following sets

$$A_K = K \cap \{u_1 > u_2\} \quad \text{and} \quad B_K = K \cap \{u_1 < u_2\},$$

where K is a compact subset of Ω. At least one of them need to have positive Lebesgue measure, otherwise $u_1 = u_2$. Assume that $\lambda(A_K) > 0$ for some $K \Subset \Omega$, where λ is the Lebesgue measure. Then it follows from Kołodziej's subsolution theorem (see e.g. [27]) that there exists a function $\psi \in \mathcal{E}_0$ such that $(dd^c\psi)^n = \chi_{A_K}(-w)^{1-p}\,d\lambda$, where χ_{A_K} is the characteristic function for A_K in Ω. Note that $\nu = \chi_A(-w)^{1-p}\,d\lambda \in \mathcal{M}_p$, and

$$\left|\nu(u)\right| = \left|\int_\Omega (u_1 - u_2)(-w)^{p-1}\,d\nu\right| = \int_A (u_1 - u_2)\,d\lambda > 0.$$

Part (2): Let $p \geqslant 1$. As in part (1) we start by constructing an injective, continuous, and linear map $S : \delta\mathcal{E}_p \ni u \to S_u \in \delta\mathcal{M}_p'$, where

$$S_u(\mu) = \int_\Omega (-w)^{p-1}(-u)\,d\mu.$$

This construction yields, in the similar way to part (1) of the proof, that $|S_u(\mu)| \leqslant \|u\|_p|\mu|_p$. In this way we identify $\delta\mathcal{E}_p$ with a subset of $\delta\mathcal{M}_p'$. The continuity of S follows in a similar manner as in part (1), and since $\mathcal{M}_p$ separates points of $\delta\mathcal{E}_p$ we obtain that S is injective. We know from the proof of part (1) that the map T is injective, and therefore $\mathcal{E}_p$ separates points of $\delta\mathcal{M}_p$.

Part (3): Let $p > 0$. Take any $u \in \delta\mathcal{E}_p$, then $u = u_1 - u_2$ for some $u_1, u_2 \in \mathcal{E}_p$. Without loss of generality we can assume that $u_1 \neq u_2$. Since the smallest harmonic majorants of u_1 and u_2 are identically 0, we have by the Riesz decomposition theorem that $\Delta u_1 \neq \Delta u_2$. Hence, there exists a non-pluripolar set $\omega \Subset \Omega$ such that

$$\int_\omega \Delta u_1 > \int_\omega \Delta u_2.$$

Then the operator $D_\omega \in \mathcal{D}$ satisfies that $D_\omega(u) \neq 0$, and this proof is completed. □

Theorem 5.4. *Let $p > 0$. Then*

(1) *$\mathcal{E}_p$ is a normal cone in $(\delta\mathcal{E}_p, \succcurlyeq, \|\cdot\|_p)$.*
(2) *$(\delta\mathcal{E}_p, \succcurlyeq, \|\cdot\|_p)^r = (\delta\mathcal{E}_p, \succcurlyeq, \|\cdot\|_p)^b = (\delta\mathcal{E}_p, \succcurlyeq, \|\cdot\|_p)'$, i.e.*

$$\left(\delta\mathcal{E}_p, \succcurlyeq, \|\cdot\|_p\right)' = \mathcal{E}_p' - \mathcal{E}_p'.$$

(3) *The vector space $(\delta\mathcal{E}_p, \succcurlyeq, \|\cdot\|_p)'$, $p \geqslant 1$, is equal to the closure of $\delta\mathcal{M}_p$ in $\sigma((\delta\mathcal{E}_p)', \delta\mathcal{E}_p)$.*
(4) *The vector space $(\delta\mathcal{E}_p, \succcurlyeq, \|\cdot\|_p)'$ is equal to the $\sigma((\delta\mathcal{E}_p)', \delta\mathcal{E}_p)$-closure of the linear span of $\mathcal{D}$.*

The correspondent statements for parts (1)–(3) are also true for $(\delta\mathcal{M}_p, \succcurlyeq, \|\cdot\|_p)$.

Proof. *Part* (1), *case* $\delta\mathcal{E}_p$: Since we can view $(\delta\mathcal{E}_p, \succcurlyeq, \|\cdot\|_p)$ as a metrizable space, it is sufficient to consider sequences of $(\delta\mathcal{E}_p, \succcurlyeq, \|\cdot\|_p)$, rather than nets. Assume that $\{u_n\}$ and $\{v_n\}$ are sequences of $(\delta\mathcal{E}_p, \succcurlyeq, \|\cdot\|_p)$ such that for all n we have that

$$u_n \succcurlyeq v_n \succcurlyeq 0, \quad \text{and} \quad u_n \overset{\|\cdot\|_p}{\to} 0. \tag{5.2}$$

Recall that the positive cone $(\delta\mathcal{E}_p)^+$ of $(\delta\mathcal{E}_p, \succcurlyeq)$ is precisely $\mathcal{E}_p$, and therefore by assumption (5.2) we can assume that $\{u_n\}, \{v_n\} \subset \mathcal{E}_p$. Then there exist $\alpha_n \in \mathcal{E}_p$ such that $u_n - v_n = \alpha_n$, and therefore by Lemma 4.4 we have that

$$\|u_n\|_p = e_p(u_n)^{\frac{1}{n+p}} = e_p(v_n + \alpha_n)^{\frac{1}{n+p}} \geqslant e_p(v_n)^{\frac{1}{n+p}} = \|v_n\|_p.$$

Hence, $v_n \overset{\|\cdot\|_p}{\to} 0$.

Part (1), *case* $\delta\mathcal{M}_p$: This follows as the $\delta\mathcal{E}_p$ case, but instead of Lemma 4.4 one uses Lemma 4.5.

Part (2), *case* $\delta\mathcal{E}_p$: Lemma 5.2 implies that $(\delta\mathcal{E}_p)^r \subseteq (\delta\mathcal{E}_p)^b \subseteq (\delta\mathcal{E}_p)'$, and therefore it remains to show that $(\delta\mathcal{E}_p)' \subseteq (\delta\mathcal{E}_p)^r$. Take a functional $T \in (\delta\mathcal{E}_p)'$, and for $u \in \mathcal{E}_p$ fixed, let us define

$$[0, u] = \{v \in \mathcal{E}_p\colon\ u \succcurlyeq v \succcurlyeq 0\}.$$

Consider the map $q : \mathcal{E}_p \to \mathbb{R}$ defined by

$$q(u) = \sup\bigl\{T(v)\colon\ v \in [0, u]\bigr\}.$$

This map has the properties that

$$q(tu) = tq(u) \quad \text{and} \quad q(u+v) \geqslant q(u) + q(v), \quad \text{for all } u \in \mathcal{E}_p,\ t \geqslant 0.$$

The inequality follows from the fact that $[0,u] + [0,v] \subseteq [0, u+v]$. Hence, the following set

$$C = \{(t,u) \in \mathbb{R} \times \mathcal{E}_p\colon\ 0 \leqslant t \leqslant q(u)\}$$

is a cone in $\mathbb{R} \times \delta\mathcal{E}_p$.

Let $(X, \|\cdot\|)$ be a quasi-Banach space such that X' separates the points of X. Then X' is a Banach space with the norm

$$\|x^*\| = \sup_{\substack{\|x\| \leqslant 1 \\ x \in X}} \left|x^*(x)\right|.$$

From $(X, \|\cdot\|)$ we can now construct the *Banach envelope* X_c of X by defining the norm

$$\|x\|_c = \sup_{\substack{\|x^*\| \leqslant 1 \\ x^* \in X'}} \left|x^*(x)\right|, \quad x \in X.$$

Then $\|\cdot\|_c$ is the largest norm on X that is dominated by the original quasi-norm $\|\cdot\|$, or in other words, $\|\cdot\|_c$ is the Minkowski functional of the convex hull of the unit ball. Furthermore, if we set $X_c = \overline{(X, \|\cdot\|_c)}$, then $X'_c = X'$.

By Lemma 5.3 we can now use this construction for $X = \delta\mathcal{E}_p$ to prove that $(1,0) \notin \bar{C}$, where the closure is taken in $\mathbb{R} \times (\delta\mathcal{E}_p)_c$. By Lemma 5.2, and the fact that all bounded linear functionals on $\delta\mathcal{E}_p$ separates points, we have that $(\delta\mathcal{E}_p)'_c = (\delta\mathcal{E}_p)'$.

Assume that $(1,0) \in \bar{C}$. Then there exists a sequence $\{(t_j, u_j)\} \subset C$ that converges in the product topology to $(1,0)$. In particular,

$$\|u_j\|_c = \sup_{\substack{\|S\| \leqslant 1 \\ S \in (\delta\mathcal{E}_p)'}} \left|S(u_j)\right| \to 0, \quad \text{as } j \to +\infty.$$

Let $u \in \mathcal{E}_p$, and for each j we define an operator by

$$S_j(u) = \begin{cases} \|u_j\|_p^{1-n-p} \int_\Omega (-u_j)^p \left(dd^c u\right) \wedge \left(dd^c u_j\right)^{n-1} & \text{if } 0 < p < 1, \\ \|u_j\|_p^{1-n-p} \int_\Omega (-u)(-u_j)^{p-1} \left(dd^c u_j\right)^n & \text{if } p \geqslant 1. \end{cases}$$

Then, $S_j \in (\delta\mathcal{E}_p)'_c$, and $\|S_j\| = 1$. Furthermore, we have that

$$\left|S_j(u_j)\right| = \|u_j\|_p \leqslant \|u_j\|_c \to 0, \quad \text{as } j \to +\infty,$$

hence $\|u_j\|_p \to 0$. Thus, for all $v \in \mathcal{E}_p$ with $u_j \succcurlyeq v$, $j \in \mathbb{N}$, we have that $u_j = v + \alpha_j$, for some $\alpha_j \in \mathcal{E}_p$ and

$$\|v\|_p = e_p(v)^{\frac{1}{n+p}} \leqslant e_p(v + \alpha_j)^{\frac{1}{n+p}} = e_p(u_j)^{\frac{1}{n+p}} = \|u_j\|_p \to 0,$$

as $j \to +\infty$. This implies, as $j \to +\infty$, that $t_j \leqslant q(u_j) \to 0$. This contradicts the assumption that $(1,0) \in \bar{C}$. Thus, $(1,0) \notin \bar{C}$.

In the next step we shall use that $(\mathbb{R} \times \delta\mathcal{E}_p)'$ and $(\mathbb{R} \oplus (\delta\mathcal{E}_p)_c)'$ are isomorphic. Hahn–Banach theorem implies that there exists $H \in (\mathbb{R} \oplus (\delta\mathcal{E}_p)_c)'$ such that $H \geqslant 0$ on C and $H(1,0) = -1$. We can write $H(t,u) = -t + g(u)$, where $g(u) = H(0,u) \geqslant 0$, since $(0,u) \in C$. For $u \in \mathcal{E}_p$ it then follows that $(q(u), u) \in C$, hence

$$H\big(q(u), u\big) = -q(u) + g(u) \geqslant 0.$$

This yields that

$$g(u) \geqslant q(u) \geqslant T(u),$$

and therefore $g - T \in \mathcal{E}'_p$. Thus, $T = g - (g - T) \in \mathcal{E}'_p - \mathcal{E}'_p = (\delta\mathcal{E}_p)^r$.

Part (2), *case* $\delta\mathcal{M}_p$: As noted in the proof of Theorem 3.2, $(\delta\mathcal{M}_p, \succcurlyeq)$ is a Riesz space, and therefore it follows immediately that $(\delta\mathcal{M}_p)^r = (\delta\mathcal{M}_p)^b$. Furthermore, by Lemma 5.2 we have that $(\delta\mathcal{M}_p)^b \subset (\delta\mathcal{M}_p)'$. It therefore remains to prove that $(\delta\mathcal{M}_p)' \subset (\delta\mathcal{M}_p)^b$. Assume that $T \in (\delta\mathcal{M}_p)'$. Then for all $\mu \in \delta\mathcal{M}_p$ we have that

$$\left|T(\mu)\right| \leqslant \|T\| |\mu|_p, \quad \text{where } \|T\| = \sup_{\substack{|\nu|_p \leqslant 1 \\ \nu \in \delta\mathcal{M}_p}} \left|T(\nu)\right|.$$

For all $\nu \in [0,\mu]$ we have that $\nu \leqslant \mu$, so by Theorem 2.2 $u_\mu \leqslant u_\nu$, where $u_\mu, u_\nu \in \mathcal{E}_p$ and $(dd^c u_\mu)^n = \mu$, $(dd^c u_\nu)^n = \nu$. Thus

$$\left|T(\nu)\right| \leqslant \|T\||\nu|_p = \|T\|\left(\int_\Omega (-u_\nu)^p\, d\nu\right)^{\frac{n}{n+p}} \leqslant \|T\|\left(\int_\Omega (-u_\mu)^p\, d\mu\right)^{\frac{n}{n+p}} = \|T\||\mu|_p.$$

Thus, $T([0,\mu])$ is bounded, i.e. $T \in (\delta\mathcal{M}_p)^b$.

Part (3), *case* $\delta\mathcal{E}_p$: Lemma 5.3 yields that $\mathcal{M}_p$ separates of $(\delta\mathcal{E}_p)'$, and hence the $\sigma((\delta\mathcal{E}_p)', \delta\mathcal{E}_p)$-closed linear span of $\mathcal{M}_p$ is $(\delta\mathcal{E}_p)'$. Thus, $(\delta\mathcal{E}_p, \succcurlyeq, \|\cdot\|_p)'$ is equal to the closure of $\delta\mathcal{M}_p$ in $\sigma((\delta\mathcal{E}_p)', \delta\mathcal{E}_p)$.

Part (3), *case* $\delta\mathcal{M}_p$: As in part (3), case $\delta\mathcal{E}_p$.

Part (4): As in part (3), case $\delta\mathcal{E}_p$. □

Remark. The spaces $(\delta\mathcal{E}_1, \|\cdot\|_1)$ and $(\delta\mathcal{M}_1, |\cdot|_1)$ are not reflexive. The proof of Proposition 6.2 in [30] works here as well.

Remark. In [16], it was proved that $(\delta\mathcal{F})' = \mathcal{F}' - \mathcal{F}'$, and it was proved in [30] that $(\delta\mathcal{E})' = \mathcal{E}' - \mathcal{E}'$.

Remark. At the first glance of Theorem 5.4 it seems possible to deduce (2) immediately from (1), by using the seminal work of Krein [29]. This is not possible unless $p = 1$, since $(\delta\mathcal{E}_p, \|\cdot\|_p)$ $(p \neq 1)$ is only a quasi-normed space.

Example 5.5 shows that $\mathcal{D} \cap \mathcal{M}_p = \{0\}$, for all $p \geqslant 1$.

Example 5.5. We shall in this example use the argument by contradiction. Let $p \geqslant 1$. Assume that there exists an element in $0 \neq D_\omega \in \mathcal{D} \cap \mathcal{M}_p$, i.e. there exist a non-pluripolar set $\omega \Subset \Omega$, $w \in \mathcal{E}_0$, and $\mu \in \mathcal{M}_p$ such that

$$D_\omega(u) = \int_\omega \Delta u = \int_\Omega (-w)^{p-1}(-u)\, d\mu \quad \text{for all } u \in \mathcal{E}_p.$$

Take z and $r > 0$ such that $B(z,r) \Subset \Omega$. Fix $u \in \mathcal{E}_p$, and let $\epsilon > 0$ be such that

$$\sup\bigl\{u(z)\colon\ z \in \omega \cup B(z,r)\bigr\} + \epsilon < 0. \tag{5.3}$$

The function defined by $v = (\sup\{w \in \mathcal{E}_p\colon\ w \leqslant u + \epsilon \text{ on } \omega \cup B(z,r)\})^*$ is in $\mathcal{E}_p$, and satisfies: $v \geqslant u$ and $v = u + \epsilon$ on $\omega \cup B(z,r)$. Hence,

$$0 = D_\omega(u) - D_\omega(v) = \int_\Omega (-w)^{p-1}(v-u)\, d\mu \geqslant 0.$$

Thus, $\mu = 0$ on $\omega \cup B(z,r)$, since $\mu(\{v > u\}) = 0$. The point z in Ω was chosen arbitrarily, and therefore we can conclude that $\mu = 0$. Thus $D_\omega = 0$ and a contradiction is obtained.

6. Modulability

Let $\mathcal{K}$ be a generating cone in a vector space X, i.e. every function u in X can be written as $u = u_1 - u_2$, $u_1, u_2 \in \mathcal{K}$. This decomposition is not unique. In Theorem 6.2, it is proved that there exists a decomposition of each elements in $\delta\mathcal{E}_p$ with explicit control of the quasi-norm.

Lemma 6.1. *Let $p > 0$, and $u, v \in \mathcal{E}_p$ be such that $v \leqslant u$. Then*

$$e_p(u) \leqslant D(n,p)^{\frac{n+p}{p}} e_p(v),$$

where $D(n,p)$ is the constant defined in Theorem 4.1. If in addition $p \leqslant 1$, then $e_p(u) \leqslant e_p(v)$.

Proof. Let $p > 0$. Then we have that

$$e_p(u) = \int_\Omega (-u)^p \bigl(dd^c u\bigr)^n \leqslant \int_\Omega (-v)^p \bigl(dd^c u\bigr)^n \leqslant D(n,p) e_p(u)^{\frac{n}{n+p}} e_p(v)^{\frac{p}{n+p}},$$

which implies that

$$e_p(u) \leqslant D(n,p)^{\frac{n+p}{p}} e_p(v).$$

For the second statement assume that $p \leqslant 1$. Then by [14] there exist decreasing sequences $\{u_j\}, \{v_j\} \subset \mathcal{E}_0$, with $u_j \geqslant v_j$,

$$u_j \to u, \qquad v_j \to v, \qquad e_p(u_j) \to e_p(u) \quad \text{and} \quad e_p(v_j) \to e_p(v) \quad \text{as } j \to +\infty.$$

For $p \leqslant 1$, the function $-(-u_j)^p$ is a plurisubharmonic function. Hence, by [13] we get that

$$e_p(u_j) = \int_\Omega (-u_j)^p \left(dd^c u_j\right)^n \leqslant \int_\Omega (-u_j)^p \left(dd^c v_j\right)^n \leqslant \int_\Omega (-v_j)^p \left(dd^c v_j\right)^n = e_p(v_j).$$

This proof is then completed by letting $j \to +\infty$. $\square$

Theorem 6.2. *For each $u \in \delta\mathcal{E}_p$ there exist uniquely determined functions $u^-, u^+ \in \mathcal{E}_p$ such that $u = u^+ - u^-$, and*

$$\|u\|_p \leqslant \left\|u^+ + u^-\right\|_p \leqslant D(n,p)^{\frac{1}{p}} \|u\|_p.$$

Furthermore, if $p \leqslant 1$, then $\|u\|_p = \|u^+ + u^-\|_p$.

Proof. Let $u = u_1 - u_2 \in \delta\mathcal{E}_p$, and define

$$u^- = \sup\{\beta \in \mathcal{E}_p\colon \text{ there exists } \alpha \in \mathcal{E}_p \text{ such that } u_1 + \beta = u_2 + \alpha\},$$

and

$$u^+ = \sup\{\alpha \in \mathcal{E}_p\colon \text{ there exists } \beta \in \mathcal{E}_p \text{ such that } u_1 + \beta = u_2 + \alpha\}.$$

Then $(u^+)^*, (u^-)^* \in \mathcal{E}_p$, where $(w)^*$ denotes the upper semicontinuous regularization of w. By the classical Choquet's lemma (see e.g. [25]) there exist sequences $\{\alpha_j\}$, $\{\beta_j\} \subset \mathcal{E}_0$ such that $(\sup_j \beta_j)^* = (u^-)^*$, and $(\sup_j \alpha_j)^* = (u^+)^*$. Furthermore, we can assume that $u_1 + \beta_j = u_2 + \alpha_j$. By passing to the limit we get that

$$u_1 + u^- = u_2 + u^+.$$

But $u^- = (u^-)^*$ and $u^+ = (u^+)^*$, outside a pluripolar set (quasi-everywhere), which means that $u_1 + (u^-)^* = u_2 + (u^+)^*$. Therefore, $u^+ \geqslant (u^+)^*$ and $u^- \geqslant (u^-)^*$. Hence,

$$u^+ = \left(u^+\right)^* \quad \text{and} \quad u^- = (u^-)^*.$$

If $\alpha, \beta \in \mathcal{E}_p$ are such that $u_1 + \beta = u_2 + \alpha$, then $\alpha \leqslant u^+$ and $\beta \leqslant u^-$. Thus, Lemmas 4.4 and 6.1 yield that

$$\|u\|_p \leqslant e_p\left(u^+ + u^-\right)^{\frac{1}{n+p}} = \left\|u^+ + u^-\right\|_p \leqslant D(n,p)^{\frac{1}{p}} e_p(u_1 + u_2)^{\frac{1}{n+p}},$$

and for $p \leqslant 1$

$$\|u\|_p \leqslant e_p\left(u^+ + u^-\right)^{\frac{1}{n+p}} = \left\|u^+ + u^-\right\|_p \leqslant e_p(u_1 + u_2)^{\frac{1}{n+p}}.$$

Taking infimum over all decomposition $u_1 - u_2 = u$ we get desired inequalities. $\square$

Remark. Theorem 6.2 is also valid for $\delta\mathcal{F}$.

In the context of normal cone, which was used in Section 5, there is the dual notion of modulability. A cone $\mathcal{K}$ in a quasi-normed space $(X, \|\cdot\|)$ is said to be *modulable* if there exists a constant $\gamma > 0$ such that

- $\mathcal{K}$ is generating, i.e. $X = \mathcal{K} - \mathcal{K}$, and
- $\gamma |||(u_1, u_2)||| \leqslant \|u\|$ for all $u_1, u_2 \in \mathcal{K}$, $u = u_1 - u_2$. Here $|||\cdot|||$ denotes the usual l^2-norm on $X \times X$, or any other equivalent norm.

This definition implies that $0 \leqslant \gamma \leqslant 1$. The terminology of modulability is not unified. Other equivalent terminology are for example non-oblateness, non-flattening, and strict b-cone. The property of being modular is well known under the condition that $(X, \|\cdot\|)$ is normed. For example any generating cone in a normed space is modular, but the converse is generally false without any extra assumptions on X (see e.g. [5,24,28]). In Corollary 6.3, we obtain an estimate of the modular constant γ for the functions u^-, u^+ in Theorem 6.2. The key ingredient of the proof is the simple fact that the function $f : (0, +\infty) \to \mathbb{R}$ defined by

$$f(q) = \left(\frac{1}{2}\left(a^q + b^q\right)\right)^{\frac{1}{q}}$$

is non-decreasing for $a, b \geqslant 0$. This means that if $0 < q_1 \leqslant q_2$, then $f(q_2) \geqslant f(q_1)$, and therefore it holds that

$$\left(a^{q_2} + b^{q_2}\right)^{\frac{1}{q_2}} \geqslant \left(\frac{1}{2}\right)^{\frac{q_2-q_1}{q_1 q_2}} \left(a^{q_1} + b^{q_1}\right)^{\frac{1}{q_1}}. \tag{6.1}$$

Furthermore, the function $g : (0, +\infty) \to \mathbb{R}$ defined by

$$g(q) = \left(a^q + b^q\right)^{\frac{1}{q}}$$

is non-increasing for $a, b \geqslant 0$. So if $0 < q_1 \leqslant q_2$, then $f(q_1) \geqslant f(q_2)$, and therefore it holds that

$$\left(a^{q_1} + b^{q_1}\right)^{\frac{1}{q_1}} \geqslant \left(a^{q_2} + b^{q_2}\right)^{\frac{1}{q_2}}. \tag{6.2}$$

Corollary 6.3. *Let $p > 0$, and $u \in \delta\mathcal{E}_p$. If $u^-, u^+ \in \mathcal{E}_p$ are as in Theorem 6.2, then*

$$\gamma\left(u^+, u^-, ||| \cdot |||_{l^2}\right) \geqslant \begin{cases} 1 & \text{if } n = 1,\ 0 < p \leqslant 1, \\ D(1,p)^{-\frac{1}{p}} & \text{if } n = 1,\ p > 1, \\ (\frac{1}{2})^{\frac{n+p-2}{2(n+p)}} & \text{if } n \geqslant 2,\ 0 < p \leqslant 1, \\ (\frac{1}{2})^{\frac{n+p-2}{2(n+p)}} D(n,p)^{-\frac{1}{p}} & \text{if } n \geqslant 2,\ p > 1, \end{cases}$$

and

$$\gamma\left(u^+, u^-, ||| \cdot |||_{l^\infty}\right) \geqslant \begin{cases} (\frac{1}{2})^{\frac{n}{n+1}} & \text{if } 0 < p \leqslant 1, \\ (\frac{1}{2})^{\frac{n}{n+1}} D(n,p)^{-\frac{1}{p}} & \text{if } p > 1. \end{cases}$$

Here $D(n, p)$ is the constant defined in Theorem 4.1.

Proof. Let $n \geqslant 1$ and $p > 0$. Then Theorem 6.2 implies that there exist $u^-, u^+ \in \mathcal{E}_p$ such that $u = u^+ - u^-$, and $\|u\|_p \geqslant D(n,p)^{-\frac{1}{p}} \|u^+ + u^-\|_p$. Furthermore, we know that $e_p(u^+ + u^-) \geqslant e_p(u^+) + e_p(u^-)$. Hence, by Lemma 4.4 we get that

$$\begin{aligned} \|u\|_p &\geqslant D(n,p)^{-\frac{1}{p}} \left\|u^+ + u^-\right\|_p = D(n,p)^{-\frac{1}{p}} e_p\left(u^+ + u^-\right)^{\frac{1}{n+p}} \\ &\geqslant D(n,p)^{-\frac{1}{p}} \left(e_p\left(u^+\right) + e_p(u^-)\right)^{\frac{1}{n+p}} = D(n,p)^{-\frac{1}{p}} \left(\left\|u^+\right\|_p^{n+p} + \|u^-\|_p^{n+p}\right)^{\frac{1}{n+p}}. \end{aligned} \tag{6.3}$$

If in addition $p \leqslant 1$, then we know by Theorem 6.2 that $\|u\|_p = \|u^+ + u^-\|_p$, and therefore we have that

$$\|u\|_p \geqslant \left(\left\|u^+\right\|_p^{n+p} + \|u^-\|_p^{n+p}\right)^{\frac{1}{n+p}}. \tag{6.4}$$

For the case $n + p \geqslant 2$, (6.1) yields that

$$\left(\left\|u^+\right\|_p^{n+p} + \|u^-\|_p^{n+p}\right)^{\frac{1}{n+p}} \geqslant \left(\frac{1}{2}\right)^{\frac{n+p-2}{2(n+p)}} \left(\left\|u^+\right\|_p^2 + \|u^-\|_p^2\right)^{\frac{1}{2}} = \left(\frac{1}{2}\right)^{\frac{n+p-2}{2(n+p)}} \left|\left|\left|\left(u^+, u^-\right)\right|\right|\right|_{l^2}. \tag{6.5}$$

Inequalities (6.3)–(6.5) imply that

$$\gamma\left(u^+, u^-, ||| \cdot |||_{l^2}\right) \geqslant \begin{cases} D(1,p)^{-\frac{1}{p}} & \text{if } n = 1,\ p > 1, \\ (\frac{1}{2})^{\frac{n+p-2}{2(n+p)}} & \text{if } n \geqslant 2,\ 0 < p \leqslant 1, \\ (\frac{1}{2})^{\frac{n+p-2}{2(n+p)}} D(n,p)^{-\frac{1}{p}} & \text{if } n \geqslant 2,\ p > 1. \end{cases}$$

Let $n = 1$, $0 < p < 1$. It follows by (6.2) that

$$\left(\left\|u^+\right\|_p^{1+p} + \|u^-\|_p^{1+p}\right)^{\frac{1}{1+p}} \geqslant \left(\left\|u^+\right\|_p^2 + \|u^-\|_p^2\right)^{\frac{1}{2}} = \left|\left|\left|\left(u^+, u^-\right)\right|\right|\right|_{l^2}.$$

Thus, $\gamma(u^+, u^-, ||| \cdot |||_{l^2}) = 1$. Now we shall estimate $\gamma(u^+, u^-, ||| \cdot |||_{l^\infty})$. Using again the properties of the functions u^+, u^-, together with inequality (6.1) we obtain the following:

$$\left(\left\|u^+\right\|_p^{n+p} + \|u^-\|_p^{n+p}\right)^{\frac{1}{n+p}} \geqslant \left(\frac{1}{2}\right)^{\frac{n}{n+1}} \left(\left\|u^+\right\|_p + \|u^-\|_p\right) \geqslant \left(\frac{1}{2}\right)^{\frac{n}{n+1}} \left|\left|\left|\left(u^+, u^-\right)\right|\right|\right|_{l^\infty}. \tag{6.6}$$

This proof is then completed by combining (6.6) with (6.3), and (6.4), respectively. □

Remark. Theorem 6.2 is valid for $\delta\mathcal{F}$, as already noted. Furthermore, we obtain the corresponding result of Corollary 6.3. We get that

$$\gamma\big(u^+, u^-, \|\!|\cdot\|\!|_{l^2}\big) \geqslant \begin{cases} 1 & \text{if } n=1, \\ (\frac{1}{2})^{\frac{n-2}{2n}} & \text{if } n \geqslant 2, \end{cases}$$

and $\gamma(u^+, u^-, \|\!|\cdot\|\!|_{l^\infty}) \geqslant (\frac{1}{2})^{\frac{n-1}{n}}$.

Corollary 6.4. *Let $p>0$, $\mu \in \delta\mathcal{M}_p$. Then*

$$\gamma\big(\mu^+, \mu^-, \|\!|\cdot\|\!|_{l^2}\big) = 1,$$

and

$$\gamma\big(\mu^+, \mu^-, \|\!|\cdot\|\!|_{l^\infty}\big) = 1.$$

Proof. This follows as the proof of Corollary 6.3, but note that for $\mu \in \delta\mathcal{M}_p$ it follows that

$$|\mu|_p = \big|\mu^+\big|_p + |\mu^-|_p \geqslant \big(\big|\mu^+\big|_p^2 + |\mu^-|_p^2\big)^{\frac{1}{2}} = \big\|\!\big|\big(\mu^+, \mu^-\big)\big\|\!\big|_{l^2},$$

and similarly

$$|\mu|_p = \big|\mu^+\big|_p + |\mu^-|_p \geqslant \big\|\!\big|\big(\mu^+, \mu^-\big)\big\|\!\big|_{l^\infty}. \qquad \Box$$

By employing a similar technique as in the proof of Theorem 6.2 we obtain, in Corollary 6.5, that $\delta\mathcal{E}_p(\Omega)$ is not separable.

Corollary 6.5. *Let $p>0$. Then*

(a) *$\delta\mathcal{E}_p(\Omega)$ is not separable, i.e. there does not exist any countable dense set in $\delta\mathcal{E}_p(\Omega)$, and*
(b) *$\delta\mathcal{M}_p(\Omega)$ is not separable.*

Proof. (a) Fix $w \in \Omega$ and let g_w be the pluricomplex Green function with pole at w. For $a<b<0$ we define

$$u_a(z) = \max\big(g_w(z), a\big) \quad \text{and} \quad u_b(z) = \max\big(g_w(z), b\big).$$

Then we have that $u_a, u_b \in \mathcal{E}_0 \cap C(\Omega)$. Assume for now that we have proved that there exists a constant $\epsilon>0$ such that for all $-2 \leqslant a, b \leqslant -1$ it holds that

$$\|u_a - u_b\|_p \geqslant \epsilon. \tag{6.7}$$

Then it follows that $\delta\mathcal{E}_p(\Omega)$ is not separable by Corollary 1.6.2 in [36]. By Theorem 6.2 we get that

$$\|u_a - u_b\|_p \geqslant D(n,p)^{-\frac{1}{p}} e_p\big((u_a-u_b)^+ + (u_a-u_b)^-\big)^{\frac{1}{n+p}}. \tag{6.8}$$

If we take any $\alpha, \beta \in \mathcal{E}_0$ such that $u_a + \beta = u_b + \alpha$, then we have that

$$\big(dd^c u_a\big)^n + dd^c u_a \wedge T + \big(dd^c \beta\big)^n = \big(dd^c u_b\big)^n + dd^c u_b \wedge S + \big(dd^c \alpha\big)^n,$$

with

$$\operatorname{supp}\big(dd^c u_a\big)^n, \operatorname{supp} dd^c u_a \wedge T \subseteq \big\{g_w(z) \geqslant a\big\} \quad \text{and}$$
$$\operatorname{supp}\big(dd^c u_b\big)^n, \operatorname{supp} dd^c u_b \wedge S \subseteq \big\{g_w(z) \geqslant b\big\},$$

where $S = \sum_{k=1}^{n-1}\binom{n}{k}(dd^c u_a)^{k-1} \wedge (dd^c\beta)^{n-k}$ and $T = \sum_{k=1}^{n-1}\binom{n}{k}(dd^c u_b)^{k-1} \wedge (dd^c\alpha)^{n-k}$. Hence, $\{g_w(z) = a\} \subseteq \operatorname{supp}(dd^c\alpha)^n$. Therefore, we get that $(dd^c\alpha)^n \geqslant (dd^c u_a)^n$. Theorem 2.2 yields that $u_a \geqslant \alpha$. If we now choose $\alpha = (u_a - u_b)^+$, and $\beta = (u_a - u_b)^-$, then we can conclude that $(u_a - u_b)^+ = u_a$. Furthermore, since $u_a + (u_a - u_b)^- = u_b + (u_a - u_b)^+$, we finally get that $(u_a - u_b)^- = u_b$. The inequality (6.8) now implies that

$$\|u_a - u_b\|_p \geqslant D(n,p)^{-\frac{1}{p}} e_p(u_a + u_b)^{\frac{1}{n+p}} \geqslant D(n,p)^{-\frac{1}{p}} \max\big(e_p(u_a)^{\frac{1}{n+p}}, e_p(u_b)^{\frac{1}{n+p}}\big).$$

Thus, for every $-2 \leqslant a, b \leqslant -1$ we have that

$$\|u_a - u_b\|_p \geqslant D(n,p)^{-\frac{1}{p}} e_p(u_{-1})^{\frac{1}{n+p}} = (2\pi)^{\frac{n}{n+p}} D(n,p)^{-\frac{1}{p}},$$

and this proof is then completed by setting $\epsilon = (2\pi)^{\frac{n}{n+p}} D(n,p)^{-\frac{1}{p}}$.

(b) This follows similarly as the first part. Let $\mu_a = (dd^c u_a)^n$. It is enough by Corollary 1.6.2 in [36] to prove that there exists a constant $\epsilon > 0$ such that for all $-2 \leqslant a, b \leqslant -1$ it holds that

$$|\mu_a - \mu_b|_p \geqslant \epsilon. \tag{6.9}$$

We have that $\operatorname{supp} \mu_a \cap \operatorname{supp} \mu_b = \emptyset$, since $\operatorname{supp} \mu_a \subset \{g_w(z) = a\}$, and $\operatorname{supp} \mu_b \subset \{g_w(z) = b\}$. Hence, $(\mu_a - \mu_b)^+ = \mu_a$ and $(\mu_a - \mu_b)^- = \mu_b$. We then obtain that

$$|\mu_a - \mu_b|_p = \|u_{\mu_a}\|_p^n + \|u_{\mu_b}\|_p^n = \big((-a)^p (2\pi)^n\big)^{\frac{n}{n+p}} + \big((-b)^p (2\pi)^n\big)^{\frac{n}{n+p}} \geqslant 2(2\pi)^{\frac{n^2}{n+p}} = \epsilon. \qquad \square$$

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